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The WKB Approximation Without Divergences

D. Cocolicchio[(1,2)] and M. Viggiano[(1)]

[1)] *Dipartimento di Matematica, Univ. Basilicata, Potenza, Italy*
Via N. Sauro, 85, 85100 Potenza, Italy

[2)] *Istituto Nazionale di Fisica Nucleare, Sezione di Milano, Italy*
Via G. Celoria 16, 20133 Milano, Italy

ABSTRACT

In this paper, the WKB approximation to the scattering problem is developed without the divergences which usually appear at the classical turning points. A detailed procedure of complexification is shown to generate results identical to the usual WKB prescription but without the cumbersome connection formulas.

I. Introduction

In general an exact solution for many quantum mechanical problems is unavoidable and one is forced to resort to some type of approximate technique. One of the most useful of those methods is the semiclassical or Wentzel-Kramers-Brillouin (WKB) approach [1]. The major shortcoming of the semiclassical WKB approximation of solving the wave equation is its divergence at the classical turning points. Presently, available regularization schemes are accurate, but rather complicated. Although these methods sharpen the threshold effects, nevertheless exact solutions to the stationary wave equation

$$\frac{d^2\psi}{dx^2} + k^2(x)\psi = 0 \tag{1}$$

(with the local wavenumber $k^2(x) = 2m[E - V(x)]/\hbar^2$) cannot be found in most problems which involve one-dimensional potential $V(x)$ [2]. In the WKB approach, the wave function $\psi(x)$ is supposed to be represented by

$$\psi(x) = Ae^{i\sigma(x)/\hbar} \tag{2}$$

which converts the linear, time-independent Schrödinger equation for $\psi(x)$ into the non-linear Riccati equation for the function $\sigma(x)$

$$\left(\frac{d\sigma}{dx}\right)^2 - i\hbar\frac{d^2\sigma}{dx^2} = 2m[E - V(x)] \quad . \tag{3}$$

The WKB approximation consists in expanding σ as a power series in $\hbar$:

$$\sigma(x) \simeq \sigma_0(x) + \frac{\hbar}{i}\sigma_1(x) + \left(\frac{\hbar}{i}\right)^2 \sigma_2(x) + o(\hbar^3) \tag{4}$$

and substituting this expression in the relative differential equation, whose coefficients can be generated recursively [3]. The following conditions can then be obtained:

$$\begin{cases} \left(\frac{d\sigma_0(x)}{dx}\right)^2 = 2m[E - V(x)] \equiv p^2(x) \\ \left(\frac{d\sigma_0(x)}{dx}\right)\left(\frac{d\sigma_1(x)}{dx}\right) + \frac{1}{2}\left(\frac{d^2\sigma_0(x)}{dx^2}\right) = 0 \end{cases} \tag{5}$$

whose solutions are given by

$$\begin{aligned}
\sigma_0(x) &= \pm \int_{x_0}^{x} p(x')\,dx' \\
\sigma_1'(x) &= -\frac{1}{2}\frac{\sigma_0{}''(x)}{\sigma_0'(x)} = -\frac{1}{2}\frac{p'(x)}{p(x)} \Rightarrow \quad \sigma_1(x) = -\ln\sqrt{p(x)}
\end{aligned} \tag{6}$$

where the prime denotes differentiation with respect to x and x_0 is an arbitrary point. The leading connection term $\sigma_1(x)$ diverges at the classical turning points $x_c^{(i)}$ where $V(x_c^{(i)}) = E$, which makes the first-order WKB solution

$$\psi(x) \simeq \frac{1}{\sqrt{p(x)}}\left\{C_+ \exp\left(\frac{i}{\hbar}\int_{x_0}^{x} p(x')\,dx'\right) + C_- \exp\left(-\frac{i}{\hbar}\int_{x_0}^{x} p(x')\,dx'\right)\right\} \tag{7}$$

divergent in these points. In the classical limit, this divergence is understandable, since a classical particle has zero velocity at these turning points. This divergence at the turning points is a severe limitation on the usefulness of the WKB approximation for quantum mechanics, since there is no divergence in the exact wave function. However away from the turning points, the WKB solution gives a good description of wave functions, especially in the semiclassical limit of large quantum numbers n. Far from the turning points, the behaviour of the semiclassical solution

$$\psi(x) = \frac{1}{\sqrt{p(x)}}\left\{C_+ \exp\left(i\int_{x_0}^{x} k(x')\,dx'\right) + C_- \exp\left(-i\int_{x_0}^{x} k(x')\,dx'\right)\right\} \tag{8}$$

changes drastically in accord with the wave-number

$$k(x) = \frac{1}{\hbar}p(x) = \frac{1}{\hbar}\sqrt{2m[E - V(x)]} \quad . \tag{9}$$

An oscillatory behaviour is produced by the solution corresponding to the classically allowed region $E > V_{max}(x)$, where $k(x)$ is real:

$$\psi(x) = \frac{A}{\sqrt{p(x)}}\sin\left\{\int_{x_0}^{x} k(x')\,dx' + \frac{\pi}{4}\right\} \tag{10}$$

whereas in the classically forbidden region $E < V_{min}(x)$ where the wave-number $k(x) = i\beta(x)$ with

$$\beta(x) = \frac{1}{\hbar}\sqrt{2m[V(x) - E]} > 0 \quad , \tag{11}$$

becomes purely imaginary, the general semiclassical solution is exponentially decrescent:

$$\psi(x) = \frac{1}{\sqrt{\beta(x)}} \left\{ C_+ \exp\left(- \int_{x_0}^{x} \beta(x')\, dx' \right) + C_- \exp\left(\int_{x_0}^{x} \beta(x')\, dx' \right) \right\} \quad . \tag{12}$$

This semiclassical behaviour is valid only asymptotically. The regions near the turning points should be treated separately. This leads to the fragmentation of the x axis into several regions with connection formulas for going through the turning points. Such patchwork for the semiclassical wave functions leads to the familiar lowest-order WKB energy quantization condition

$$\int_{x_c^{(1)}}^{x_c^{(2)}} \sqrt{2m[E_n - V(x)]}\, dx = \left(n + \frac{1}{2} \right) \pi\hbar \tag{13}$$

or to more complicated conditions if higher orders corrections in $\hbar$ are kept [4]. On the other side, continous connection formulas giving finite wave functions also at the turning points have also been developed in the realm of uniform approximation [5]. In the next section we recover the connection formulas of the usual WKB procedure, then in a later section we establish that this prescription can be generalized with a reformulation based on the analytic continuation into the complex momentum variable and contour integrations.

II. Semiclassical Approach to Barrier Penetration

In the case of scattering problems, there are two independent solutions which are usually called incoming and outgoing waves, respectively. These waves become free-particle waves in the asymptotic region. Classically a particle (incident from the left with energy $E < V_{max}(x)$) is completely reflected from the potential region at the (left-hand) classically turning point a, defined by

$$V(a) = E \quad . \tag{14}$$

However, quantum mechanically the particle can "tunnel" through the barrier and find itself on the right-hand side. In the case of ordinary barrier penetration $V_{min} < E < V_{max}$ there exist two classical turning points on the real axis $x_c^{(1)} = a < x_c^{(2)} = b$. The WKB approximation is valid where transmission dominates over reflection and, in the semiclassical limit, the probability of tunneling is given by $T = e^{-2\sigma_*}$ where

$$\sigma_* = \int_a^b \beta(x)\, dx = \frac{1}{\hbar} \int_a^b \sqrt{2m[V(x) - E]}\, dx \tag{15}$$

and the probability of being reflected is correspondingly reduced from its classical value of unity to $R = 1 - T$. The problem of the breakdown of the WKB solution near the turning points can be overcome with the usual procedure to consider the linear approximation of the potential

$$V(x) = V(a) + \mu(x-a) \tag{16}$$

with

$$\mu = \left(\frac{dV}{dx}\right)_{x=a} \tag{17}$$

and

$$p(x) = \sqrt{2m\mu(x-a)} \quad . \tag{18}$$

Thus, the connection formulas, which relate oscillatory and exponential behaviour of the wavefunction forms on the opposite sides of a classical turning point, can be matched by considering the differential equation

$$\left(\frac{d^2}{dz^2} - z\right)\psi(z) = 0 \tag{19}$$

once we set $z = \left(2m\mu/\hbar^2\right)^{1/3}(x-a)$. The complex method for treating classical turning points provides a powerful description to the "connection problem" by using the exact solution of the Schrödinger's equation

$$\psi(x) = \frac{1}{\sqrt{p(x)}}\left\{C_+(x)\exp\left(\frac{i}{\hbar}w(x_0,x)\right) + C_-(x)\exp\left(-\frac{i}{\hbar}w(x_0,x)\right)\right\} \tag{20}$$

where

$$w(x_0,x) = \int_{x_0}^{x} p(x')\,dx'. \tag{21}$$

The usual WKB method, by its very nature, cannot take into systematic account the modifying effects of the multiply reflections between the classical turning points. Even if the values of the WKB multipliers $C_\pm(x)$ are known on a given wide region, the complex method, while providing useful information about the solutions, it results inefficient to give all the details of the wave function in the neighbourhood of the turning points. A powerful technique to overcome this deficiency consists in deriving the asymptotic solutions of the differential Airy equation (19) calculated in the stationary phase approximation. Its solution can be represented by the Laplace integral

$$\psi(x) = A\int_C e^{zt - t^3/3}\,dt \tag{22}$$

where the curve C is taken so that the integrand vanishes at the limit of integration. Since the integrand vanishes exponentially in the following interval

$$|arg\,t| < \frac{\pi}{6} \tag{23}$$

$$\frac{\pi}{2} < arg\,t < \frac{5}{6}\pi \tag{24}$$

$$\frac{7}{6}\pi < arg\,t < \frac{3}{2}\pi \tag{25}$$

the curves C_1, C_2, C_3 are all allowed and yield three different solutions and two each are independent. The integration along imaginary axis C_1 yields the so called Airy function

$$Ai(z) = \frac{1}{2\pi i}\int_{C_1} \exp\left(zt - \frac{t^3}{3}\right) dt = \frac{1}{\pi}\int_0^{+\infty} \cos\left(zt + \frac{t^3}{3}\right) dt \tag{26}$$

whereas the integration along a complementary curve C_2 yields a further independent solution

$$Bi(z) = \frac{1}{2i}\int_{C_2} \exp\left(zt - \frac{t^3}{3}\right) dt = \int_0^{+\infty} \sin\left(zt + \frac{t^3}{3}\right) dt \tag{27}$$

whose well-known asymptotic behaviours are given by

$$Ai(z) \sim \frac{1}{2\pi z^{1/4}}\exp\left(-\frac{2}{3}z^{3/2}\right) \tag{28}$$

$$Ai(-z) \sim \frac{1}{\pi z^{1/4}}\sin\left(\frac{2}{3}z^{3/2} + \frac{\pi}{4}\right) \quad . \tag{29}$$

Similarly in the case of $Bi(z)$ we have

$$Bi(z) \sim \frac{1}{z^{1/4}}\exp\left(\frac{2}{3}z^{3/2}\right) \tag{30}$$

$$Bi(-z) \sim \frac{1}{z^{1/4}}\cos\left(\frac{2}{3}z^{3/2} + \frac{\pi}{4}\right) \quad . \tag{31}$$

These two Airy functions are independent solutions of the Eq.(19). If we introduce the variable

$$\tau = \frac{2}{3}(-z)^{3/2} \tag{32}$$

and make the transformation

$$\psi(z) = (-z)^{1/2}\phi(\tau) \tag{33}$$

Eq.(19) takes the Bessel form

$$\tau^2 \frac{d^2\phi}{d\tau^2} + \tau \frac{d\phi}{d\tau} + \left(\tau^2 - \frac{1}{9}\right)\phi(\tau) = 0 \tag{34}$$

which let us represent Airy functions in terms of the following Bessel functions

$$Ai(z) = \frac{\sqrt{z}}{3}\left\{I_{-1/3}\left(\frac{2}{3}z^{3/2}\right) - I_{1/3}\left(\frac{2}{3}z^{3/2}\right)\right\} \tag{35}$$

$$Bi(z) = \sqrt{\frac{z}{3}}\left\{I_{-1/3}\left(\frac{2}{3}z^{3/2}\right) + I_{1/3}\left(\frac{2}{3}z^{3/2}\right)\right\} \quad . \tag{36}$$

The formal connection formulas can be established by the analytic features of the potential barriers near a classical turning point to match approximate solutions across the boundaries. The derivation and application of the connection formulas are both non trivial and fraught with pitfalls associated with the existence of exponentially large and exponentially small components of the wave function, in the classically forbidden region. However in spite of these difficulties, in the case $z > 0$ ($dV/dx > 0$) we have the local wavenumber in the forbidden region

$$\beta(x) = \frac{1}{\hbar}\sqrt{2m[V(x) - E]} = \frac{1}{\hbar}\sqrt{2m\mu(x-a)} \tag{37}$$

so that

$$\int_a^x \beta(x')\,dx' = \frac{2}{3}z^{3/2} \quad . \tag{38}$$

Similarly for $z < 0$ we have in the allowed region

$$k(x) = \frac{1}{\hbar}\sqrt{2m[E - V(x)]} = \frac{1}{\hbar}\sqrt{-2m\mu(x-a)} \tag{39}$$

and

$$\int_x^a k(x')\,dx' = \frac{2}{3}(-z)^{3/2} \quad . \tag{40}$$

Finally, since $\sin\left(\phi + \frac{\pi}{4}\right) = \cos\left(\phi - \frac{\pi}{4}\right)$ we can derived the following matching expression

$$\frac{2}{\sqrt{k(x)}}\cos\left(\int_x^a k(x')\,dx' - \frac{\pi}{4}\right) \longleftrightarrow \frac{1}{\sqrt{\beta(x)}}\exp\left(-\int_a^x \beta(x')\,dx'\right) \quad . \tag{41}$$

Since $\cos\left(\phi+\frac{\pi}{4}\right) = -\sin\left(\phi-\frac{\pi}{4}\right)$ we find

$$\frac{1}{\sqrt{k(x)}}\sin\left(\int_x^a k(x')\,dx' - \frac{\pi}{4}\right) \longleftrightarrow -\frac{1}{\sqrt{\beta(x)}}\exp\left(\int_a^x \beta(x')\,dx'\right) \quad . \tag{42}$$

The connection relations for the case of a decreasing potential ($dV/dx < 0$) are given analogously

$$\frac{1}{\sqrt{\beta(x)}}\exp\left(-\int_x^a \beta(x')\,dx'\right) \longleftrightarrow \frac{2}{\sqrt{k(x)}}\cos\left(\int_a^x k(x')\,dx' - \frac{\pi}{4}\right) \tag{43}$$

and

$$-\frac{1}{\sqrt{\beta(x)}}\exp\left(\int_x^a \beta(x')\,dx'\right) \longleftrightarrow \frac{1}{\sqrt{k(x)}}\sin\left(\int_a^x k(x')\,dx' - \frac{\pi}{4}\right) \quad . \tag{44}$$

With these connection formulae, it is straightforward to determine the transmission coefficient T with the knowledge of the semiclassical wave function

$$\psi(x) = \frac{1}{\sqrt{k(x)}}\left(C_+ e^{i\sigma_*} + C_- e^{-i\sigma_*}\right) \tag{45}$$

and its asymptotic limit

$$\psi(x) \sim \begin{cases} \psi_{Iin}(x) & \text{for } x \ll a \\ \psi_{IIIout}(x) & \text{for } b \ll x \end{cases} \tag{46}$$

in the region to the far left and right of the barrier, which are given by

$$\begin{aligned} \psi_{Iin}(x) = -\frac{B}{\sqrt{k(x)}}[&\exp\left(-\int_a^b \beta(x)dx\right)\sin\left(\int_x^a k(x')dx' - \frac{\pi}{4}\right) + \\ &+4i\exp\left(\int_a^b \beta(x)dx\right)\sin\left(\int_x^a k(x')dx' + \frac{\pi}{4}\right)] \end{aligned} \tag{47}$$

and

$$\psi_{IIIout}(x) = \frac{2B}{\sqrt{k(x)}}\exp\left(i\int_b^x k(x')dx' - i\frac{\pi}{4}\right) \tag{48}$$

let us make an immediate determination of the transmission coefficient T. This probability of tunneling is defined by means of the ratio between the probability current density

$$j = Re\left(\frac{\hbar}{im}\psi^*\frac{d\psi}{dx}\right) = -i\frac{\hbar}{2m}\left(\psi^*\frac{d\psi}{dx} - \psi\frac{d\psi^*}{dx}\right) \tag{49}$$

of the transmitted and the incident waves:

$$T = \frac{j_{III}}{j_I} \tag{50}$$

where

$$\begin{cases} j_{III} = 4|B|^2 \frac{\hbar}{m} \\ j_I = 4|B|^2 \left[\left(e^{\sigma_*} + \frac{1}{4}e^{-\sigma_*}\right)^2 - \left(e^{\sigma_*} - \frac{1}{4}e^{-\sigma_*}\right)^2 \right] \frac{\hbar}{m} \end{cases} \tag{51}$$

with

$$\sigma_* = \int_a^b \beta(x)\, dx = \frac{1}{\hbar}\int_a^b \sqrt{2m[V(x) - E]}\, dx \quad . \tag{52}$$

Thus the WKB techniques show that in the semiclassical limit, the probability of tunneling is given by

$$T = \frac{j_{III}}{j_I} = \frac{e^{-2\sigma_*}}{\left(1 + \frac{1}{4}e^{-2\sigma_*}\right)^2} \simeq e^{-2\sigma_*} \tag{53}$$

that is valid under the assumption that $\exp(-2\sigma_*) \ll 1$. Therefore the probability of being reflected is correspondingly reduced to $R = 1 - T$.

III. Effective Semiclassical Approximation to Barrier Penetration

In the complex method we previously analyzed, no attempt is made to clarify the contributions of the multipliers $C_\pm(x)$ which indeed are x-dependent and associated to the modifying effects of the possibility of internal reflections inside the barrier, in the sense that the particle can travel from turning point a to turning b in a several n number of ways. A systematic account of multiply reflected contributions in the limit of a continuous potential is associated with the following coupled first-order equations

$$C'_\pm(x) = {\sigma'}_1(x) C_\mp(x) \exp\left(\mp \frac{2i}{\hbar} w(x_0, x)\right) \tag{54}$$

where the prime refers to the derivation with respect to x [6]. These coupled equations are formally equivalent to the Schrödinger's Eq.(1). The different solutions of the Schrödinger's equation are obtained by applying different conditions to $C_\pm(x)$. In particular, imposing that there is no reflected wave far beyond the potential barrier and assuming that the incident wave has unit intensity, we obtain the scattering solution, after a trivial change of normalization,

$$\begin{cases} C_+(x) = 1 + \int_{-\infty}^{x} dx'\, r(x') C_-(x') \exp\left[-\frac{2i}{\hbar} w(x_0, x')\right] \\ C_-(x) = -\int_{x}^{+\infty} dx'\, r(x') C_+(x') \exp\left[+\frac{2i}{\hbar} w(x_0, x')\right] \quad . \end{cases} \tag{55}$$

Wherever the differential reflection coefficient

$$r(x) = -\sigma_1'(x) = \frac{p'(x)}{2p(x)} \tag{56}$$

may be set equal to zero, a constant value for the $C_\pm(x)$ are obtained, as we mentioned above. Therefore, the precise value of the reflection coefficient may be found with the help of successive integration by parts

$$\begin{aligned} R = -\int_{-\infty}^{+\infty} dx\, r(x) C_+(x) \exp\left[+\frac{2i}{\hbar} w(x_0, x)\right] = -\int_{-\infty}^{+\infty} dx\, r(x) \exp\left[+\frac{2i}{\hbar} w(x_0, x)\right] - \\ -\int_{-\infty}^{+\infty} dx\, r(x) \left\{\int_{-\infty}^{x} dx'\, r(x') C_-(x') \exp\left[-\frac{2i}{\hbar} w(x_0, x')\right]\right\} \exp\left[\frac{2i}{\hbar} w(x_0, x)\right] \quad . \end{aligned} \tag{57}$$

In this case, however, the main contributions to the integral cannot be selected easily because they depend on the analytic properties of the function $p(x)$ and, in the last view, on the type of the singularities of the potential. In fact, the Schrödinger Eq.(1) takes the form

$$\frac{d^2\phi(w)}{dw^2} + \left[\frac{1}{\hbar^2} + \widetilde{V}(w)\right]\phi(w) = 0 \tag{58}$$

where we change the variable to the phase w of the exponent in Eq.(54) and make the transformation

$$\phi(x) = \sqrt{\frac{p(x)}{\hbar}}\psi(x) \tag{59}$$

being $\widetilde{V}$ determined by p through the relation

$$\widetilde{V} = \frac{3(p')^2 - 2pp''}{4p^4} = \frac{\sigma_1'' + {\sigma_1'}^2}{{\sigma_0'}^2} = -\frac{2}{p(x)}\sigma_2' \quad . \tag{60}$$

Here prime denotes differentiation with respect to x. Nevertheless, we may stress that the most important quantity results the phase w of the exponent. The subtleties involved in the evaluation of the precise value of the reflection coefficient stem on the convergent expansion of the total wave amplitude which can be given in momentum space as

$$\begin{aligned} \widetilde{\psi}(k_0, k) = \widetilde{\phi}(k_0, k) + \int dk'\, \widetilde{G_0}(k_0, k')\widetilde{V}(k_0, k')\widetilde{\phi}(k_0, k') + \\ + \int dk'' \left[\int dk'\, \widetilde{G_0}(k_0, k)\widetilde{V}(k_0, k')\widetilde{G_0}(k', k'')\widetilde{V}(k', k'')\widetilde{\phi}(k', k'')\right] + ... \end{aligned} \tag{61}$$

where $\widetilde{\phi}$ represents the free wave solution and

$$\widetilde{G}_0(k) = \frac{\hbar^2}{2\pi}\left[\frac{1}{1-(\hbar k)^2}\right] \tag{62}$$

is its relative propagator. Both ϕ_0 and $\widetilde{G_0}$ are obtained assuming that the perturbation $\widetilde{V}$ vanishes in Eq.(58). Using these asymptotic solutions, we derive that the exact solution of reflection coefficient is perturbatively given by

$$\begin{aligned} R = \widetilde{v}(k_i, k_f) + &\int dk_1\, \widetilde{v}(k_i, k_1)\widetilde{G}_0(k_1)\widetilde{v}(k_1, k_f) + \\ &+ \int\!\!\int dk_1 dk_2\, [\widetilde{v}(k_i, k_1)\widetilde{G}_0(k_1)\widetilde{v}(k_1, k_2)\widetilde{G}_0(k_2)\widetilde{v}(k_2, k_f)] + ... \end{aligned} \tag{63}$$

where $\widetilde{v}(k, k')$ are the matrix elements of the perturbation $\widetilde{V}$ in Eq.(58)

$$\begin{aligned} \widetilde{v}(k, k') &= \int d\xi \widetilde{\phi}^*(k)\widetilde{V}(w(\xi))\widetilde{\phi}(k') \\ &= \int_{-\infty}^{+\infty} e^{\frac{i}{\hbar}(k'-k)\xi}\widetilde{V}(w(\xi))\, d\xi \quad . \end{aligned} \tag{64}$$

Such results would need to take account of all the singularities of the perturbation $\widetilde{V}$ and indeed of $p(x)$. Thus any value of R will not in general be single valued because of the branch points at the real turning points $x_c^{(i)}$ where $p(x_c^{(i)}) = 0$, unless we adopt the convention of dividing the complex Gauss plane into two Riemann sheets as shown in Fig.(1).

Concluding Remarks

The propagator (Green's function) technique in the solution of problems in non relativistic Quantum Mechanics becomes relevant in calculating explicitly the reflection coefficient in the case of barrier scattering represented by an analytic function $V(x)$. In the case of ordinary barrier penetration $E < V_{max}$, $k(x)$, defined in Eq.(9), is in general two-sheeted, with two branch turning points $x_c^{(1)}$, $x_c^{(2)}$ located on the real axis and we may choose the defining branch cut to connect them (Fig. 1). An alternative approach to the semiclassical approximation allows a very appealing picture of the transmission coefficient and generates results identical to the usual WKB prescription, but without the cumbersome connection formulas. This method consists in the analytic continuation into the complex momentum variable and contour integrations, wherein it is permitted. Such a method of complex variable in modern theoretical physics is extensively adopted to clarify the concepts of analyticity in S-matrix theory. It results particularly suitable

for discussing the problem of one dimensional barrier penetration if we reconsider that the generic propagation from x_1 to x_2 far to the left and right of the barrier respectively in opposite sides can be considered as occurring in three successive steps according to the decomposition of the outstanding integral

$$\left(\int_{x_{in}}^{x_{fin}} k(x)\,dx\right) = \left(\int_{x_{in}}^{a} k(x)\,dx + \int_{a}^{b} i\beta(x)\,dx + \int_{b}^{x_{fin}} k(x)\,dx\right) \quad . \tag{65}$$

The complexification of the problem generates a more intuitive picture of the physics of the process and offers an alternative technique to avoid singularities which occur in the standard WKB. The propagator can be, then, expressed in terms of contour integrals which connects the initial and final points x_{in}, x_{fin}, both of which are located on the real axis far to the left of the barrier. Furthermore we assert that if the incident wave is initially located in x_{in} far to the left of the barrier, then the reflected wave is given by the analytic continuation of the functions involved, evaluated on the other side of the cut. It is important to note that they are on different sheets so that any independent contour of integration has to pass through the cut. Of course, the singularities of the function $\widetilde{V}$ in the complex plane may involve other branch points, but we may discard them here. Anyway, the singularities of $\widetilde{V}$ are related to the singularities of the function $p(x)$. Clearly, the multiple integrals resulting in Eq.(63) correspond to the effect of multiple reflections. The contributions of the only once–reflected waves to the reflection coefficient are given by

$$R \simeq -\int_{-\infty}^{+\infty} dx\, r(x) e^{2iw(x_0,x)/\hbar} \quad . \tag{66}$$

Actually, one could include also contributions from contours which loop the branch cut several times. These higher order terms are, in general, unreliable although they are expected exponentially much smaller than the polynomial corrections which lie outside the usual semiclassical WKB approximation.

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